October 2010
Version – 5

Essential Mathematics for Undergraduate Students in Engineering

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Please forward corrections to the author. Thank you in advance.

Engineering majors require a strong foundation in engineering mathematics in order to become comfortable with fundamental principles in various core courses. By the end of the sophomore year, most students should have completed the equivalent of four semesters of college level math, culminating in concepts associated with solution of ordinary differential equations and linear algebra principles. Often, students find it difficult to make the connection between the basic mathematics taught in the math courses and the math required as a tool for engineering courses. The purpose of this document is to highlight the relevant math concepts in the context of engineering courses. This single document focuses on the most important mathematical concepts/jargon that you will require to enable you to ***study*** math and engineering textbooks on your own. A complete mastery of the material discussed here is expected of all undergraduate students entering the junior year; you should be familiar with most concepts by the middle of the sophomore year. It is suggested that you review this material each semester so that the concepts become second nature to you; *learn to interpret and commit all formulae to memory – the most important ones are highlighted*. It will also serve you well if you intend to pursue graduate studies in engineering. To fully comprehend the material and to see worked out examples you will of course have to refer to any math book that discusses these concepts in more detail.

Here is a listing of topics discussed in this document.

A) The Derivative
B) Principle of Dimensional Consistency
C) Differentials
D) Chain Rule
E) Polynomials
F) Taylor Series
G) Multivariable Functions and Partial Derivatives
H) Simplest Ordinary Differential Equations
I) Integration
J) Grad, Div, Curl, and all that
K) Elementary Linear Algebra
L) Complex Numbers and Complex Variables

A. The Derivative

Given a function $y = f(x)$, sometimes denoted as $y(x)$, the *derivative,* denoted by $y'(x)$ or $f'(x)$ is formally defined by the limit:

$$\boxed{\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x}} \qquad (1)$$

Note that the engineering units of the derivative will be the units of *y* divided by the units of *x*. Thus if *y* has the units of energy in kcal, and *x* has the units of time in seconds, the quantity $y'(x)$ will have the units of power in kcal/sec. For simplicity, we will use the notation, [*y*] to denote the dimensions of, or units of, *y*. Thus, read

$$[y] \text{ as "dimensions of } y\text{" or "units of "}y\text{"} \qquad (2)$$

B. Principle of Dimensional Consistency

It is always useful to know the dimensions of every quantity so that in an equation, one can check if indeed "apples are added to apples" and not to "oranges". Consider the algebraic equation:

$s = ut + \frac{1}{2}at^2$ that you used in physics. Here *s* denotes distance, *u* is the initial particle velocity, *a* is the acceleration and *t* is time. Every term, namely, the quantities *s*, *ut*, and ½ at^2 all have to have dimensions of length. The *standard dimensions* are length (*L*), time (*t*), mass (*M*), Force (*F*), and temperature (*T*). Units refer to the actual engineering basis for measuring these quantities. Thus the dimension length is measured in units of meters, time in seconds, mass in kilograms, force in Newtons, and temperature in degrees Kelvin, etc. If for instance you have forgotten the exponent associated with *t* in the second term, i.e., you cannot recall if it is "*t* squared" or "*t* cubed," then the principle of dimensional consistency is especially handy. Working with SI units, *s* being distance has to be in meters. The quantity *u* is in m/s and *t* is in seconds, thus *ut* has units of m. Next, note that the coefficient ½ does not have units or is a dimensionless constant. Since *a* denotes acceleration, it has units of m/s^2. This indicates that *t* has to have the exponent 2, so that at^2 ends up with units of meters. This kind of sanity check applies to all equations – algebraic, ordinary, or partial differential equations. We will refer to this principle as the ***principle of dimensional consistency*** in equations that arise in all engineering courses.

> *The principle of dimensional consistency requires that, the dimension of every term in an equation, be it, algebraic, or differential, has to be identical. In addition, arguments of transcendental functions (those that cannot be expressed as equivalent finite number of algebraic functions, example exp, sin, cosine, log, etc.) that appear in an equation have to be dimensionless.*

Get into the habit of using this principle and it will serve you well in engineering practice. It will allow you spot errors in equations. It is a necessary, <u>but not sufficient</u> condition, to carry out meaningful engineering calculations. Apply this principle to the Bernoulli equation that arises in the study of fluid mechanics. It states: $p + \frac{1}{2}\rho V^2 + \rho gh = \text{constant}$ or $\frac{P}{\rho g} + \frac{V^2}{2g} + h = \text{constant}$,

where p is the static fluid pressure, ρ is fluid density, V is velocity, g is acceleration due to gravity and h is vertical height. In the first form, all terms have dimensions of pressure (force per unit area), while in the second form, all terms have dimensions of length.

C. Differentials

Often you will come across expressions involving a *differential* quantity such as:

$$dy = \frac{dy}{dx}dx \tag{3}$$

in which it appears at first glance that the dx's have cancelled each other out. A more useful way to think about the meaning of (3) is:

change in y (or dy) = (rate of change of y with respect to x) times (change in x) (4)

One can extend this notion and write

$$y(x+dx) = y(x) + \frac{dy}{dx}dx \tag{5}$$

which states that the value of y at $x+dx$ is given by its value at x (the first term on the right hand side of (5)) plus (the rate of change of y with respect to changes in x) times (change in x, namely dx). Note that dimensions of all terms in (5) also work out correctly. Can you figure out $y(x+dx/2)$ and $y(x-dx/2)$? Simply replace "dx" in (5) by "+dx/2" or "-dx/2" to get the result. Application of this concept is usually tied to a neatly labeled sketch, in which the axis (x-axis in this case) is clearly indicated, with the arrowhead denoting the direction of increasing x.

The *second derivative* of y with respect to x is denoted as $\frac{d^2y}{dx^2}$ or $y''(x)$. The prime notation is useful for first, second, and perhaps third derivative, but is often not used beyond that. If it is, for example, the *n-th derivative* of y, it is denoted by $y^{(n)}(x)$.

What are the dimensions of $\frac{d^2y}{dx^2}$? Answer: $\left[\frac{d^2y}{dx^2}\right] = \frac{[y]}{[x]^2}$. If y has units of meters, and x is in seconds, the quantity $\frac{d^2y}{dx^2}$ denotes acceleration with units of m/s^2.

D. Chain Rule

If $y = g(f(x))$, then the derivative

$$\frac{dy}{dx} = \frac{dg}{dx} = \frac{dg}{df}\frac{df}{dx} \tag{6}$$

Note that the quantity $\frac{dg}{df}$ denotes the rate of change of g with respect to changes in f or the rate of change of g with respect to changes in its *argument*. The quantity g has f as its argument, while the quantity f has x as its argument. One can of course extend this notion and thus if $y = g(h(f(x)))$, then

$$\frac{dy}{dx} = \frac{dg}{dh}\frac{dh}{df}\frac{df}{dx} \tag{7}$$

Note that $\frac{dy}{dx}$ denotes the rate at which y changes with respect to changes in x. A quick check of dimensions in (7) reveals that:

$$\frac{[y]}{[x]} = \frac{[y]}{[h]}\frac{[h]}{[f]}\frac{[f]}{[x]} \tag{8}$$

Note that the dimensions of h and dimensions of f cancel out on the right hand side of (8) as they appear in both the numerator and denominator, leaving the dimensions of y in the numerator and that of x in the denominator. Master the concept of chain rule and it will serve you well in engineering/physics courses.

E. Polynomials

The simplest *polynomial* is a constant, also considered as a polynomial of *degree* 0. Thus,

$$P_0(x) = c \tag{9}$$

The next, in terms of simplicity, is a linear function of x. It can be written as

$$P_1(x) = c_0 + c_1 x \tag{10}$$

where c_0 and c_1 are constants, referred to as *coefficients*.
A polynomial of degree n is given by

$$P_n(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_{n-1} x^{n-1} + c_n x^n \tag{11}$$

It can be written compactly using the "sigma" or summation notation

$$P_n(x) = \sum_{k=0}^{n} c_k x^k \tag{12}$$

where k is a *dummy index* that ranges from 0 through n. It does not matter if we used any other symbol for the dummy index. Thus, on the right hand side of (12), we could use any index such as i, j, l, etc., but not n. Note that $P_n(x)$ is characterized by (n+1) constant coefficients, c_0, c_1, c_2, …c_n. Pay attention to the notation – c_m denotes the coefficient of the term containing x^m. The nice feature exhibited by polynomials, is that they remain polynomials when differentiated with respect to x. They are also easy to evaluate, differentiate, and integrate. It is easy to evaluate their roots, i.e., values of x at which $P_n(x) = 0$. Programs such as MATLAB have very simple commands to work with and manipulate polynomials and perform these tasks for you. Two polynomials are equal only *iff* (i.e., if and only if) they are of the same degree and if all of their corresponding coefficients are equal to each other. What will be the degree of the derivative of a polynomial of degree n? Can you work out what its "c_0, c_1, c_2, …c_{n-1}" are and how they relate to the "c_0, c_1, c_2, …c_n" associated with the original polynomial?

There is a special polynomial known as the **Taylor polynomial** which approximates a general function (not necessarily a polynomial) $f(x)$. It has the property that the values of the polynomial and its first n derivatives at a given point $x = a$, are identical to the values of the function f and its first n derivatives at the same point $x = a$ (try expressing this sentence through a set of n+1 equations). It is given by:

$$P_n(x) = \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!}(x-a)^m = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{13}$$

Notice that the dummy index (*m* in this case) with summation notation is a compact way of writing the polynomial. Note that

$$P_n(a) = f(a), P_n'(a) = f'(a), P_n''(a) = f''(a), \cdots, P_n^{(n)}(a) = f^{(n)}(a) \tag{14}$$

The difference between $f(x)$ and $P_n(x)$ is called the remainder and is given by

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \tag{15}$$

where ξ is an unknown number that lies in the interval (a, x). Since the $(n+1)^{\text{th}}$ derivative of a polynomial of degree n is zero (check this with simple examples that you construct on your own), the remainder is *identically zero* (meaning it is zero for all values of x, not just at a single value of x) in the case of a Taylor polynomial written for function that is a polynomial of degree n. For sufficiently smooth functions (simply put, for functions that can be differentiated again and again without any of its derivatives blowing up when evaluated at $x = a$), the remainder becomes vanishingly small as the number n approaches infinity. This leads us to the concept of a Taylor series

F. Taylor Series

The *Taylor series* of a function $f(x)$ in the neighborhood of a point $x = a$, is an infinite series given by:

$$f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(a)}{m!}(x-a)^m = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots \tag{16}$$

To clarify, *note that obtaining an expression for the Taylor series of a function in the neighborhood of a point x = a does not mean set x to a, but instead it means the series will contain infinite powers of (x-a).* A special case of the Taylor series is the **Maclaurin series**, which is the Taylor series expansion in the neighborhood of the point $x = 0$. It is given by (setting $a = 0$ in (16))

$$f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!}(x-0)^m = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \cdots \frac{x^n}{n!}f^{(n)}(0) + \cdots \tag{17}$$

The Taylor series concept is very useful in developing numerical methods for differentiation, integration, and polynomial approximation. These form the foundations of computational algorithms used to solve ordinary and partial differential equations. The Taylor series of a

function is unique, which implies that if an infinite power series (i.e., involving only positive integer powers of (x-a)) is obtained for a function via another method, then you have its Taylor series! Learn to obtain the Taylor series approximations in the neighborhood of $x = 0$ for $\exp(x), \sin(x), \cos(x)$. What if the neighborhood is around $x = \pi$? Which of these is an *even/odd* function of x? Can you figure that out by examining the Taylor series? Note that a function is even if switching the sign of its argument leaves its value unchanged, it is odd, if switching the sign of its argument causes the sign of the function to switch, while leaving its magnitude unchanged. Put in mathematical terms,

$$f(x) \text{ is } \begin{cases} \text{even if } f(t) = f(-t) \\ \text{odd if } f(t) = -f(-t) \end{cases} \tag{18}$$

Note that the argument is denoted as t to make the point that the argument here is a dummy variable that can be denoted by any symbol. The Taylor series for $f(x + \Delta x)$ in the neighborhood of x, is useful and is readily obtained by replacing x by $x + \Delta x$, and a by x in the general formula (16). The result is:

$$\begin{aligned} f(x+\Delta x) = {} & f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2!} f''(x) + \frac{\Delta x^3}{3!} f'''(x) + \\ & \frac{\Delta x^4}{4!} f^{(4)}(x) + \cdots + \frac{\Delta x^n}{n!} f^{(n)}(x) + HOT's \end{aligned} \tag{19}$$

where *HOT's* stands for 'Higher Order Terms'. Note the pattern in the series expression. Consider (5), in light of this discussion pertaining to Taylor series. It represents a *first order* approximation to (19) (the order refers to the power of Δx in the terms retained).

G. Multivariable Functions and Partial Derivatives

When a function f depends on more than one independent variable, say two independent variables x and y. Thus,

$$f = f(x, y) \tag{20}$$

Then it has two first partial derivatives, denoted by $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ (read as "partial f partial x" and "partial f partial y" or more precisely "partial derivative of f with respect to x" and "partial derivative of f with respect to y"), wherein it is understood that only changes with respect to x (or y) are considered while keeping the other variable y (or x) fixed. Thus, in evaluating $\frac{\partial f}{\partial x}$, y is simply treated as a constant (note that it is not a constant as such in general). Its formal definition is:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} \tag{21}$$

Note the dimensions: $\left[\frac{\partial f}{\partial x}\right] = \frac{[f]}{[x]}$

Three second derivatives exist – they are $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}$. In the case of the last one, the order of differentiation doesn't matter (i.e., one can differentiate with respect to y first and then x or the

other way around) if both mixed partial derivatives are continuous. This is generally the case in engineering applications. An alternate notation that you should be familiar with is the subscript notation for partial derivatives. Thus

$f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}, f_{xx} = \frac{\partial^2 f}{\partial x^2}, f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, f_{xxyy} = \frac{\partial^4 f}{\partial x^2 \partial y^2}$. Clearly the subscript notation is more compact! How many third derivatives exist assuming all derivatives involved are continuous? Answer: four. What are the dimensions of f_{xxxy} ? Answer: $\left[f_{xxxy}\right] = \frac{\left[f\right]}{\left[x\right]^3\left[y\right]}$.

Just as a Taylor series for a function of a single independent variable, may be obtained one can construct a Taylor series for a function of two (or more) independent variables. This is however, not discussed in this document, since it arises less frequently in undergraduate level engineering courses.

The concept of a *total differential* is extremely useful for functions of two or more independent variables. We will assume two independent variables (a concept that occurs in thermodynamics for example when one works with intensive properties of simple compressible substances) and that will suffice for most engineering purposes. Consider a function:

$$z = z(x, y), \tag{22}$$

in which *x* and *y* are the independent variables. The quantity *z* could denote density, pressure, fluid velocity component or other variables of interest, and *x* and *y* could denote position in two dimensional space or two intensive properties such as internal energy and specific volume. The key concept to note is that *z* can change as a result of changes in *x* or changes in *y* or due to changes in both *x* and *y*. This notion is expressed succinctly by taking the total differential of (22) which yields:

$$\boxed{dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy} \tag{23}$$

This arises time and again and can be confusing (compare (23) with (3) and note the similarity). If however, you think of it in terms of the answer to the question, what can change *z*, it is a bit easier. The first term on the right hand side of (23) denotes the change that results in *z* due to changes in *x* alone, while keeping *y* fixed. Let us break this down further. After all $\frac{\partial z}{\partial x}$ denotes the rate at which *z* changes due to changes in *x*. When this result is multiplied by the change in *x*, expressed by *dx*, the result $\frac{\partial z}{\partial x} dx$ expresses the change in *z* solely due to changes in *x*, ignoring anything to do with *y*. Using the same reasoning, the second term on the right hand side of (23) may be interpreted – it expresses the change that results in *z* due to changes in *y*, while keeping *x* fixed. Adding the two contributions gives the total change or the differential *dz*. The value of *z* at *x*+*dx*/2 and *x*-*dx*/2 are thus given by

$$z + \frac{\partial z}{\partial x}\frac{dx}{2} \text{ and } z - \frac{\partial z}{\partial x}\frac{dx}{2} \tag{24}$$

An example of (23) is illustrated by the Gibbs equation in thermodynamics

$$Tds = du + pdv \tag{25}$$

in which T denotes absolute temperature, p denotes pressure, s, u, and v denote the entropy, internal energy, and specific volume respectively. Rewrite this in true differential form by dividing (25) by T. The result is

$$ds = \frac{1}{T}du + \frac{P}{T}dv \tag{26}$$

Equation (26) can be cast in the form of (23) and interpreted as such by rewriting as

$$ds = \frac{\partial s}{\partial u}du + \frac{\partial s}{\partial v}dv \tag{27}$$

while noting that $\frac{\partial s}{\partial u} \equiv \frac{1}{T}, \frac{\partial s}{\partial v} \equiv \frac{p}{T}$ actually define (absolute) temperature and pressure formally. You will learn more about this in a thermodynamics class (usually applicable for ME/AE/ChE majors).

H. Simplest Ordinary Differential Equations

Once you have a thorough knowledge of derivatives and partial derivatives, you have the essential tools to get to the next level and learn about differential equations. We will keep this discussion simple. We will not deal with partial differential equations in which partial derivatives appear. We will instead deal with equations in which only ordinary derivatives appear. Such equations are called *ordinary differential equations* (*ODE*'s for short). There is only one independent variable (often distance or time in ME courses), while more than one dependent variable may exist (example: velocity and acceleration). The simplest ODE is a *linear, first order ordinary differential equation* of the form:

$$\boxed{\frac{dy}{dx} = -\lambda y} \tag{28}$$

Its solution is

$$\boxed{y(x) = y(0)\exp(-\lambda x)} \tag{29}$$

This may be obtained by *separating variables* in (28) and integrating:

$$\int dy / y = -\lambda \int dx \tag{29a}$$

and taking exponentials to obtain the solution given by (28). At $x = 0, y(x = 0) = y(0)$ is an initial condition. A more intuitive method is to think about a function, which when differentiated, yields back the same function (as in (28), ignoring the "$-\lambda$" for the moment). The exponential function has this property. For positive values of λ, the solution displays exponential decay, decreasing as x increases and approaching zero asymptotically for large values of x. The quantity λ is usually a physically significant quantity and, since the quantity λx in (29), being the argument of a transcendental function (exponential in this case), ought to be dimensionless according to the principle of dimensional consistency, we expect the dimension $[\lambda] = \left[\frac{1}{x}\right]$. If x denotes time (in which case the symbol t is used for the independent variable, instead of x), λ

will have dimensions of 1/time, and thus $1/\lambda$ with dimensions of time will be a characteristic time constant for the system in question. This kind of an equation arises in analysis of lumped heat transfer systems for example. It can also arise in problems associated with population growth ($\lambda > 0$), radioactive decay, drug elimination within the blood stream, etc. Note that if the right hand side of (28) is a general function of *y*, say $f(y)$, rendering the ODE nonlinear, it can be still solved by separating variables, provided the integral $\int dy / f(y)$ can be readily obtained by methods of integration or by the use of a table of integrals.

Another equation that arises often in engineering courses (vibrations, electrical circuits, wave type problems, etc) is a *linear, second order ODE with constant coefficients*:

$$\boxed{\frac{d^2y}{dx^2} = -\lambda^2 y} \tag{30}$$

It has two linearly independent solutions $y = \sin \lambda x$ and $y = \cos \lambda x$. Note that the sine and cosine functions when differentiated twice, yield themselves (with a minus sign) and thus provide a hint to the solution to (30). The complete solution is obtained by combining these solutions:

$$\boxed{y = C_1 \cos \lambda x + C_2 \sin \lambda x} \tag{31}$$

where C_1 and C_2 are two constants of integration (recall a second order ODE will require two conditions) that are determined by suitable conditions in the problem. To get this solution formally, the substitution $y = \exp(mx)$ is made in (30). After canceling out $\exp(mx)$ from every term, the resulting characteristic equation $m^2 + \lambda^2 = 0$ is solved for *m*. Then using a basic property of complex numbers:

$$\exp(i\phi) = \cos\varphi + i \sin\phi, \text{ where } i \equiv \sqrt{-1}, \tag{31a}$$

the result (31) is obtained. Complex numbers and variables are discussed in Section L. of this document.

Note: If the right hand side of (30) were $-\lambda y$, then the sine and cosine solutions will have $\sqrt{\lambda}$ in the argument making the expressions clumsy. If *x* denotes a spatial variable, the principle of dimensional consistency requires λx in (31) to be dimensionless, indicating that λ should have dimensions of 1/length or a wavenumber. If *x* denotes time, λ may be interpreted as a frequency (circular frequency would be $\lambda/(2\pi)$).

Here is a word about commonly used terminology. Obtaining a solution to an ODE may be referred to as integrating the equation, although one may not actually carry out an integration as is done to obtain tables of integrals of functions. Finally, if a problem can be reduced to an ODE, with suitable initial and/or boundary conditions, then the problem is considered (practically) solved. This is because, we have access to programs such as MATLAB or MATHEMATICA that can be routinely used to obtain a numerical solution to any desired accuracy.

I. Integration

Recall that it is easy to show that the derivative of the product of two functions $u(x)$ and $v(x)$ is given by:

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \tag{32}$$

Multiplying (32) by *dx* and rearranging gives

$$d(uv) = udv + vdu; \quad udv = d(uv) - vdu \tag{33}$$

Integrating (33) gives the formula for *integration by parts* that comes in very handy:

$$\boxed{\int udv = uv - \int vdu} \tag{34}$$

The concept of average value of a function over a range of the independent variable is important. The average value of $f(x)$ over the interval (*a*,*b*) is defined by

$$\boxed{\bar{f} \equiv \frac{1}{(b-a)}\int_a^b f(x)\,dx} \tag{35}$$

If $f(x)$ represents hydrodynamic pressure as a function of depth from a free surface, the average pressure is $\bar{f}$ (note that $\bar{f}$ will have dimensions of force/unit area or pressure. If *f* denotes heat transfer coefficient associated with fluid flow over a flat plate, $\bar{f}$ denotes an average heat transfer coefficient over the entire length of the plate. Note the dimensional consistency in (35). The *dx* and *(b-a)* terms have the same dimensions, so that $[\bar{f}] = [f]$.

In some applications (ex: computing forces on planar submerged surfaces – concepts relevant to Civil/Mechanical/Aerospace majors), you will come across the first and second moments of an area. Imagine an *x*-*y* coordinate system and a planar area located in this plane. If *dA* denotes a differential portion of this planar area (so that the integral of *dA* will give the total area *A*), then the quantity *x dA* (with dimensions of $L\ L^2 = L^3$) integrated over the entire area gives the *first moment of the area* about the *y*-axis (this is because *x* is the distance of the area *dA* from the *y*-axis) and is denoted by:

$$I_y = \int x dA \tag{36}$$

(you can best understand this by drawing a sketch, labeling the axes, and locating the distance of a portion of the area *dA* from the *y* axis). The *second moment of this area* is given by

$$I_{yy} = \int x^2 dA \tag{37}$$

Cross moments, ex: I_{xy} may be defined. Likewise, one can define I_x and I_{xx}. It should be evident that the *I*'s with a single subscript have dimensions of L^3, while the *I*'s with two subscripts will have dimensions of L^4.

J. Grad, Div, Curl, and all that

In engineering and physics, we often come up with the notion of a *field variable.* These are variables such as temperature, strain, pressure, etc., which are scalar quantities, and velocity, strain, shear stress, etc., which are either vectors or tensors in general. Think of a vector as a quantity that has both magnitude and direction, whereas a *tensor is a quantity that has magnitude and two directions*. Often, the second direction denotes the direction of the area associated with the tensor. *Treating field variables as continuous functions of position and time* allows for application of general principles to differential elements and to thereby obtain partial differential equations that describe a conservation law.

Given a *scalar* field such as temperature, the *gradient of the scalar field* is a vector given by

$$\boxed{\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}} \text{ (to be read as "Grad f")} \qquad (38)$$

written in a rectangular Cartesian coordinate system (RCCS). The quantities $\hat{i}, \hat{j}, \hat{k}$ denote the unit vectors along the three mutually perpendicular coordinate directions *x, y, z*. Note, while *f* is a scalar, ∇f is a vector. In general ∇f is the most compact way of writing down the gradient function, since this symbol is fine with any coordinate system (spherical, cylindrical, etc.), while the right hand side of (37) is specific to a RCCS. Most textbooks can be consulted to look up the specific form of this vector operator for other coordinate systems. Rules for differentiation carry over to the gradient function. Thus,

$$\nabla(fg) = f\,\nabla g + g\nabla f \qquad (39)$$

Note that $[\nabla f] = [f]/L$ as required by the principle of dimensional consistency.

Given a vector field $\vec{F}(x, y, z) = \hat{i}P(x, y, z) + \hat{j}Q(x, y, z) + \hat{k}R(x, y, z)$ such as velocity in a RCCS, the *divergence of the vector field* is given by

$$\boxed{\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}} \text{ (to be read as "div F")} \qquad (40)$$

Note that *grad of a scalar field is a vector*, whereas, *div of a vector field is a scalar*.

The *curl of a vector field* is also a vector field. It is convenient at this point to think of the symbol ∇ (to be read as "del") as an operator, which in a RCCS is given by:

$$\boxed{\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}} \qquad (41)$$

An operator, such as ∇ by itself has no meaning, but it acquires physical meaning when it operates on a quantity. The temperature gradient in a medium for instance is indicative of directions along which energy transfer in the form of heat transfer occurs. The curl of the vector $\vec{F}(x, y, z) = \hat{i}P(x, y, z) + \hat{j}Q(x, y, z) + \hat{k}\,R(x, y, z)$ is defined by

$$\boxed{\text{curl}\, F = \nabla \times \hat{F}} \tag{42}$$

which may be obtained by evaluating the determinant of the 3×3 matrix in the case of a RCC:

$$\nabla \times \hat{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \hat{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \hat{j}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \hat{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \tag{43}$$

What are the dimensions of div and curl of a vector function? Note that determinants are reviewed in a Section K of this document.

Two integral theorems arise in engineering courses. The first is the *divergence theorem* according to which the volume integral of the divergence of a vector function *F* is equal to the surface integral of *F* dotted with the surface area vector of the surface enclosing the volume in question. This provides a handy means of converting a volume integral (meaning a quantity that depends on every point within a three dimensional volume) to a surface integral (which depends remarkably only on the properties of *F* at the surface bounding this volume, without regard to the detailed variation of *F* through the three dimensional space occupied by the volume). In mathematical terms,

$$\boxed{\iiint_V \nabla \cdot F \, dV = \iint_{S \text{ bounding } V} \vec{F} \cdot d\vec{A} = \iint_{S \text{ bounding } V} \vec{F} \cdot \hat{n} \, dA} \tag{44}$$

Both *F* and $\nabla \cdot F$ are presumed to be continuous over the surface and the volume. Note that the general convention in engineering is that *area can be treated as a vector with the direction normal to the area* and assigned positive sense in the direction outward from the surface. In (44), $\hat{n}$ denotes the unit normal vector to the surface. Analogous to (44) that connects a volume integral to a surface integral, *Stokes's theorem* relates a line integral over a closed curve *C* to a surface integral over a surface *S* that is bounded by *C*.

$$\boxed{\int_C \vec{F} \cdot d\vec{l} = \iint_S (\nabla \times F) \cdot d\vec{A} = \iint_S (\nabla \times F) \cdot \hat{n} \, dA} \tag{45}$$

Note that the principle of dimensional consistency holds for (44) and (45), and is particularly handy when you can recall most of the formula and need help to figure out the exact result.

K. Elementary Linear Algebra

Most computations in engineering will involve a system of linear algebraic equations that are solved most efficiently using computing tools such as MATLAB. Two kinds of problems arise:

Type 1:

$$A\vec{x} = \vec{b} \tag{46}$$

where *A* is a square matrix of size $n \times n$, $\vec{b}$ is a column vector of size $n \times 1$ referred to as the right hand side or *forcing* vector. The unknown is a column vector $\vec{x}$ (also of size $n \times 1$), referred to as

the *solution* vector. The goal is to find $\vec{x}$. This kind of a problem arises very frequently in engineering.

Type 2:

$$A\vec{x} = \lambda\vec{x} \text{ or written alternately as } (A - \lambda I)\vec{x} = 0 \tag{47}$$

where *A* is a square matrix of size $n \times n$, *I* is an identity matrix of the same size as *A* (i.e., *I* has 1's on the diagonal and zeros elsewhere), λ is an *eigenvalue* and the corresponding solution vector is referred to as an *eigenvector*. Note that the right hand side of (47) is strictly a column vector with all zero entries. The goal is to find the eigenvalues and corresponding eigenvectors. This type of problem arises less frequently, but is equally important for all engineering majors.

Determinant of a Matrix:

In either case, it is important to have a thorough understanding of properties of a matrix. Review concepts of matrix addition and matrix multiplication on your own. Also note that the transpose of matrix, denoted by A^T, is obtained by swapping the rows and columns of *A*, and a symmetric matrix is such that $A = A^T$. The most important property that is discussed here is that of a determinant, defined only for square matrices. In the case of a 2×2 matrix,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{48}$$

the determinant is defined by:

$$\boxed{\det A \equiv |A| \equiv ad - bc} \tag{49}$$

This notion can be extended to matrices of size $n \times n$. For a general matrix *A*:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & a_{n,n-1} & a_{nn} \end{pmatrix} \tag{50}$$

The determinant is obtained by the method of cofactors. The *cofactor* of the *i*-th row and *j*-th column, denoted by M_{ij} is a square matrix of size $(n-1)\times(n-1)$ that is derived from *A*, by discarding its *i*-th row and *j*-th column. Using this definition, the determinant of *A* is obtained by:

$$\boxed{\det A = |A| = \sum_{i \text{ or } j} (-1)^{i+j} |M_{ij}|} \tag{51}$$

where the summation is carried out by either summing across a row (sum over *i* from 1 to *n*) or down a column (sum over *j* from 1 to *n*). It is interesting to note that repeated application of (51) is involved in computing the determinant of the cofactor matrices. Eventually, it will involve finding the determinant of a 2×2 matrix which is defined by (49). Applying (51) to the 3×3 matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{52}$$

by expanding using cofactors across the first row $(i = 1)$

$$\begin{aligned} \det A = {} & (-1)^{1+1} a_{11}\left(a_{22}a_{33} - a_{32}a_{23}\right) + (-1)^{1+2} a_{12}\left(a_{21}a_{33} - a_{31}a_{23}\right) + \\ & (-1)^{1+3} a_{13}\left(a_{21}a_{32} - a_{31}a_{22}\right) = \\ & a_{11}\left(a_{22}a_{33} - a_{32}a_{23}\right) - a_{12}\left(a_{21}a_{33} - a_{31}a_{23}\right) + a_{13}\left(a_{21}a_{32} - a_{31}a_{22}\right) \end{aligned} \tag{53}$$

or by expanding using cofactors down the first column $(j = 1)$

$$\begin{aligned} \det A = {} & (-1)^{1+1} a_{11}\left(a_{22}a_{33} - a_{32}a_{23}\right) + (-1)^{2+1} a_{21}\left(a_{12}a_{33} - a_{32}a_{13}\right) + \\ & (-1)^{3+1} a_{31}\left(a_{12}a_{23} - a_{22}a_{13}\right) = \\ & a_{11}\left(a_{22}a_{33} - a_{32}a_{23}\right) - a_{21}\left(a_{12}a_{33} - a_{32}a_{13}\right) + a_{31}\left(a_{12}a_{23} - a_{22}a_{13}\right) \end{aligned} \tag{54}$$

It is easy to verify that both approaches give the same numerical value for det *A* – in other words (53) and (54) are identical. Thus if a certain row or column has a number of zero entries, it makes sense to expand using cofactors for that row or column. Can you figure out why $\det A^T = \det A$? The concept of determinant is also helpful in finding the inverse of a square matrix, denoted by A^{-1}, such that $A^{-1}A = I$. Finally, a *singular matrix* is one whose determinant is zero. In the case of a 2×2 matrix, the determinant can be interpreted as the area of a suitably defined parallelogram, while for larger sized matrices, the determinant does not have a useful physical meaning.

Solution to Type 1 Problems

Consider (46)

$A\vec{x} = \vec{b}$

The solution is obtained by *Cramer's rule,* according to which, the *k*-th component of the vector $\vec{x}$, viz., x_k is obtained as:

$$\boxed{x_k = \frac{D^k}{\det A}}, \text{ where, } \boxed{D^k \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{1,k-1} & b_1 & a_{1,k+1} & \cdots & a_{1,n} \\ a_{21} & a_{22} & a_{23} & \cdots & b_2 & a_{2,k+1} & \cdots & a_{2,n} \\ a_{31} & a_{32} & a_{33} & \cdots & b_3 & a_{3,k+1} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & a_{n,k-1} & b_n & a_{n,k+1} & \cdots & a_{n,n} \end{vmatrix}} \tag{55}$$

Note that D^k is the determinant of a matrix derived from *A* in which its *k*-th column is replaced by the forcing vector. Master this method by applying it to matrix equations up to size 4×4. Matrix equations of higher dimensions are solved efficiently using MATLAB using methods that are computationally more efficient than Cramer's rule.

Solution to Type 2 Problems

Consider (47)

$A\vec{x} = \lambda\vec{x}$ or $\left(A - \lambda I\right)\vec{x} = 0$

As in the case of Type 1 problems, we will only discuss solution methods that work for small values of *n* (about 2 or 3), but become unwieldy for larger values of *n*. More efficient methods exist for finding eigenvalues and eigenvectors for larger size matrices that we will not worry about. To find the eigenvalues, find the characteristic equation by setting

$$\boxed{\det(A-\lambda I)=0} \tag{56}$$

This will be a polynomial of degree *n*. Refer to the section on polynomials to learn more about them. Using each of the computed eigenvalues λ_k, one can solve for the corresponding eigenvectors

$$A\vec{x}^{(k)} = \lambda_k \vec{x}^{(k)} \tag{57}$$

By setting an arbitrary, non-zero value for one of the components of $\vec{x}^{(k)}$, the others can be found through the remaining set of consistent algebraic equations. Although the eigenvalues are unique, the eigenvectors are not – for instance an eigenvector corresponding to an eigenvalue can be scaled by multiplying it by a real constant, and it will still be an eigenvector. Usually, eigenvectors are reported in their normalized form. A *normalized vector* is obtained by dividing each of its components by the *length* or *norm* of the vector. The norm of a vector $\vec{x}$ is defined as $\sqrt{\sum_i x_i^2}$. Thus, the norm of a normalized vector is unity. Most computer programs report eigenvectors in normalized form. Note that in the case of a diagonal matrix (meaning $a_{ij}=0, i\neq j$) or a triangular matrix, the eigenvalues are simply the entries on the diagonal.

Eigenvalues usually have a physical interpretation. In spring-mass systems subject to free (not forced) oscillations, the square root of an eigenvalue refers to the physical (angular) frequency of an oscillatory mode. The eigenvector may be interpreted as relative positions of the masses during oscillation in that particular mode. In solid mechanics, the stress tensor in three dimensions may be cast in the form of a 3×3 matrix that is symmetric. By choosing an appropriate coordinate system, the matrix can be transformed into a diagonal matrix whose entries refer to the principal components of stress. They are the eigenvalues of the stress tensor matrix. The eigenvectors are related to the direction cosines of the corresponding coordinate system.

L. Complex Numbers and Variables

The first time most students encounter complex numbers is in finding roots of a quadratic equation. Consider the algebraic equation:

$$ax^2+bx+c=0 \tag{58}$$

The *roots* ,or solutions, *x* that satisfy (58) are:

$$\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}} \tag{59}$$

When the *discriminant* "b^2-4ac" is less than zero, the roots are complex. For example $\sqrt{-49}=\sqrt{-1}\sqrt{49}=7i$, where *i* is defined by $\sqrt{-1}$ (see (31a)). Note electrical engineers might use the symbol *j* instead of *i*.

A *complex number z* is denoted by

$$z = x + i\,y \tag{60}$$

in which x and y are real numbers, where x is denoted as the *real part,* $x = \mathrm{Re}(z)$, and y its *imaginary part,* $y = \mathrm{Im}(z)$. The *conjugate* of z (z as defined in (60)), denoted by $\bar{z}$, is defined as

$$\boxed{\bar{z} = x - i\,y} \tag{61}$$

Note that when one root of a quadratic equation is complex, the other root is its complex conjugate, and thus roots of a quadratic (or for that matter a higher degree polynomial) always appear as complex conjugate pairs. The *magnitude* and *argument* of the complex number z, denoted by $|z|$ and $\arg(z)$ respectively are defined by

$$\boxed{|z| \equiv \sqrt{x^2 + y^2} \quad ; \quad \arg(z) \equiv \tan^{-1}\left(\frac{y}{x}\right)} \tag{62}$$

With these basic ideas, one can use the *Argand diagram* to represent a complex number in the x-y plane by the coordinate point (x, y). Just like negative real numbers are represented by extending the real line (or axis) to the left of zero, complex numbers can be represented by a plane in which the x and y axes are referred to as the real and imaginary axes. Thus $|z|$, which is always positive, denotes the distance r of the point z from the origin, and $\arg(z)$ denotes the angle θ, measured counterclockwise with respect to the positive x-axis, made by the line connecting the origin to z in the complex plane. In fact noting form this construction that

$$\boxed{x = r\cos\theta} \text{ and } \boxed{y = r\sin\theta} \tag{63}$$

the complex number z is denoted by its alternate *polar form*:

$$\boxed{z = x + iy = (r\cos\theta) + i\,(r\sin\theta) = r\,(\cos\theta + i\sin\theta)} \tag{64}$$

Two complex numbers z_1 and z_2 can be added via

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \tag{65}$$

in which the real and imaginary parts are added and grouped accordingly. They can be multiplied to readily obtain:

$$z_1 z_2 = (x_1 + i\,y_1)(x_2 + i y_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2) \tag{66}$$

in which the main thing to note is that, by definition $i^2 = -1$. Division is a bit tricky, but becomes easy when you learn to multiply numerator and denominator by the conjugate of the denominator to yield:

$$\frac{z_1}{z_2} = \frac{x_1 + i\,y_1}{x_2 + i\,y_2} = \frac{x_1 + i\,y_1}{x_2 + i\,y_2} \times \left(\frac{x_2 - i y_2}{x_2 - i y_2}\right) = \frac{(x_1 x_2 + y_1 y_2) + i\,(y_1 x_2 - y_2 x_1)}{(x_2^2 + y_2^2)} \tag{67}$$

wherein multiplication rules provided by (66) are used in the numerator and denominator. It is easy to separate the real and imaginary parts of the final result in (67). *Do not* commit this formula to memory, instead, note the trick that was used. In polar form, multiplication and division become quite simple. Thus,

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \times r_2\,(\cos\theta_2 + i\sin\theta_2) = r_1 r_2\,(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \tag{68}$$

and

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2\,(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)) \tag{69}$$

in which trigonometric identities are used in (68) and (69), and the trick to perform division, using the complex conjugate of the denominator to multiply both numerator and denominator, is used to get (69).

The basic algebra associated with complex numbers will serve you well for most purposes in undergraduate classes. Advanced concepts require the idea of complex variables and complex functions. Only the essential ideas are introduced here. A complex function $w = f(z)$ is given by

$$w = u + iv = f(z) = u(x, y) + iv(x, y) \tag{70}$$

in which $u(x, y)$ and $v(x, y)$ are real and imaginary parts of the function, and are themselves, functions of two variables *x* and *y*. Note the subtle aspect of (70), in which the independent variable *z* in $w = f(z)$ is linked to the real and imaginary parts via *x* and *y*. Also note that (70) represents a mapping from the *x*-*y* or *z* plane to the *u*-*v* or *w* plane. This does not lend itself to a figure, quite as easily as functions of two independent variables in calculus of real functions. Perhaps the most important concept is that of an *analytic function* which we will not define formally. In very simple terms, it is a function that is well behaved without any *singularities*. Thus $w = z, w = z^2 - 3z$ are analytic functions, while $w = 1/z$ is analytic everywhere except at $z = 0$ where it is singular, and $1/[(z-3)(z+2i)]$ has singularities at $z = 3, z = -2i$. The exponential function is very important and is defined as:

$$\boxed{w = \exp(z) = e^{x+iy} \equiv \exp(x)(\cos y + i \sin y)} \tag{71}$$

and is analytic everywhere in the complex plane. It is defined in such a manner so that it reduces to the familiar exp(*x*) when the imaginary part of *z*, viz., *y* is identically zero. Without getting into details, it has the property that

$$\frac{d}{dz}\exp(z) = \exp(z) \tag{72}$$

which is desirable when the imaginary part of *z* is zero and the result holds for exp(*x*). Similarly cos (*z*) and sin (*z*) are defined so that they become the familiar cosine and sine functions of real variables that we are familiar with when the imaginary part of *z* is identically zero.

$$\boxed{\cos z \equiv \frac{\exp iz + \exp(-iz)}{2}; \sin z \equiv \frac{\exp iz - \exp(-iz)}{2i}} \tag{73}$$

Defined in this manner, all trigonometric identities that you are familiar with, work fine for their complex counterparts (example: $\cos^2 z + \sin^2 z = 1$, etc.).

Finally, analytic functions are analogous to continuously differentiable functions in calculus of real functions. Real, continuous functions are generally useful in regions where this behavior holds, i.e., the function is continuous and differentiable, and not very useful at locations or regions where the function is "singular" or blows up. However, complex functions are actually most useful at locations where they are singular! When you work with vibrations or control systems (either as a ME/AE/EE major), you will find that identification of singularities of a complex function and the consequent behavior of the system that is being modeled are closely connected. To study fluid flow around an airfoil (ME/AE), the flow field is constructed by a suitable combination of complex functions with singularities distributed in such a manner as to generate the shape of the airfoil.